REGULAR PRECESSION OF A FREE GYROSTAT

(REGULIARNAIA PRETSESSIA SVOBODNOGO GIROSTATA)

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This note deals with questions related to certain motions of a free gyrostat in a central Newtonian field of forces.

Let O be the origin of a fixed Cartesian system of coordinates ξ_1, ξ_2 and ξ_3 at the center of gravitation, and let a moving system of coordinates x_1, x_2 and x_3, with unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$, the axes of which coincide with the principal central axes of inertia of the gyrostat, be rigidly attached to this gyrostat. Also, in the following we shall require, an orbital system of coordinates in the form of a trihedron, defined by the position vector of the mass center of the system, and by the transversal and binormal to the orbit. The unit vectors of this system will be denoted by $\mathbf{j}_1$, $\mathbf{j}_2$ and $\mathbf{j}_3$. Finally, when analysing the motion of a mechanical system relative to the mass center, we shall always resort to the Koenig system of axes ξ_1', ξ_2' and ξ_3'.

Let A_1, A_2 and A_3 denote the principal central moments of inertia of the gyrostat assumed to be a rigid body, and let M be its total mass. The moment of momentum of the gyrostat, consisting of a carrier S and gyrostatic elements g is, relative to O, expressed by [1]

$$\mathbf{K}_0 = \mathbf{R} \times M\mathbf{V} + \mathbf{K}, \qquad \mathbf{K} = \mathbf{K}_1 + \mathbf{k} \qquad (R^2 = \xi_1^2 + \xi_2^2 + \xi_3^2)$$

Here $\mathbf{R}$ is the position vector of the system's center of mass, $\mathbf{V}$ is its velocity, $\mathbf{K}$ is the moment of momentum of the gyrostat in its motions relative to the mass center, $\mathbf{K}_1$ is the moment of momentum of the system considered as a single rigid body, and $\mathbf{k}$ is the moment of momenta relative to S.

If ω_1, ω_2 and ω_3 are the projections of the instantaneous angular velocity vector ω of body S on the moving axes x_1, x_2 and x_3, then the projections of vector $\mathbf{K}_1$ on the same axes will be $A_1\omega_1$, $A_2\omega_2$ and $A_3\omega_3$. The projections of vector $\mathbf{k}$ will be denoted by k_1, k_2 and k_3.

With the above notations the equations of motion of an arbitrary gyrostat, moving in a central Newtonian field of forces defined by function U, will be

$$M\frac{d^2\xi_i}{dt^2} = \frac{\partial U}{\partial \xi_i} \qquad (i = 1\ 2\ 3) \tag{1.1}$$

$$A_1 \frac{d\omega_1}{dt} + \frac{dk_1}{dt} + (A_3 - A_2)\,\omega_3\omega_2 + \omega_2 k_3 - \omega_3 k_2 = L_1 \tag{1.2}$$

$$(A_1\, A_2\, A_3;\quad k_1\, k_2\, k_3;\quad \omega_1\, \omega_2\, \omega_3;\quad L_1\, L_2\, L_3)$$

The remaining two equations in each of the above groups are obtained by a cyclic transposition of variables appearing in parantheses; L_1, L_2 and L_3 denote the moments of Newtonian forces, acting on the system, relative the respective axes.

We introduce the following notations for the direction cosines of the moving axes with respect to the fixed axes, and to the axes of the orbital coordinate system

	x_1	x_2	x_3
ξ_1	α_{11}	α_{12}	α_{13}
ξ_2	α_{21}	α_{22}	α_{23}
ξ_3	α_{31}	α_{32}	α_{33}

	x_1	x_2	x_3
y_1	τ_{11}	τ_{12}	τ_{13}
y_2	τ_{21}	τ_{22}	τ_{23}
y_3	τ_{31}	τ_{32}	τ_{33}

The cosines of the first group above are absolute, while those of the second one are relative, and satisfy the following expressions

$$\tau_{11} = \alpha_{11}\frac{\xi_1}{R} + \alpha_{21}\frac{\xi_2}{R} + \alpha_{31}\frac{\xi_3}{R}, \qquad \tau_{12} = \alpha_{12}\frac{\xi_1}{R} + \alpha_{22}\frac{\xi_2}{R} + \alpha_{32}\frac{\xi_3}{R}$$
$$\tau_{13} = \alpha_{13}\frac{\xi_1}{R} + \alpha_{23}\frac{\xi_2}{R} + \alpha_{33}\frac{\xi_3}{R} \tag{1.3}$$

For the force function we have the known expression [2]

$$U = \frac{\mu M}{R} - \frac{3}{2}\frac{\mu}{R}(A_1\tau_{11}^2 + A_2\tau_{12}^2 + A_3\tau_{13}^2) + \frac{3}{2}\frac{\mu}{R^3}\frac{A_1 + A_2 + A_3}{3} \tag{1.4}$$

Then,

$$L_1 = \frac{3\mu}{R}(A_3 - A_2)\,\tau_{13}\tau_{12} \qquad (A_1\; A_2\; A_3;\ \tau_{11}\; \tau_{12}\; \tau_{13})$$

It remains to add to the system of equations (1.1) the Poisson's kinematic equation, and the equations of relative motions, i.e. equations which define the mechanical aspects of motions of the gyrostatic elements g, and thus complete the system of equations determining the motion of an arbitrary gyrostat in a central Newtonian field of forces.

2. We shall consider a gyrostat of the gyroscopic type, i.e. such for which the central ellipsoid of inertia is an ellipsoid of revolution. We denote by A_1 and A_3 its equatorial and axial moments ofinertia respectively.

Let the inner motion be represented by a symmetric rotor, in steady rotation, the axis of which is stationary with respect to the carrier S, and is directed along the axis of symmetry of the gyrostat. There is no friction in the rotor bearings which exercise on its axis normal forces only.

In this case

$$k_1 = k_2 = 0, \qquad k_3 = k = \text{const}, \qquad \mathbf{K} = A_1\omega_1\mathbf{i}_1 + A_1\omega_2\mathbf{i}_2 + (A_3\omega_3 + k)\,\mathbf{i}_3$$

The gravitational moment, created by the field, relative to the mass center is expressed by

$$\mathbf{L}\,(L_1, L_2, L_3) = \frac{3\mu}{R}(A_3 - A_1)\,\tau_{13}\,(\mathbf{j}_1 \times \mathbf{i}_3) \tag{2.1}$$

The expression 'regular precession' of a free gyrostat will be used here to describe a motion in which the carrier rotates with a constant angular velocity about the axis of symmetry x_3 of the gyrostat, while x_3 in turn rotates with a constant angular velocity ω^* about another axis ξ_3', passing through the center of the mass system, and having a fixed position in space.

It can be shown that the moment L of external forces, provided it is not zero, and under the condition that the gyrostat of the described kind is subject to regular precession, must have the form

$$L = (H - A_1\,\omega^*\cos\varphi_2)\,\omega^*\times \mathbf{i}_3 \qquad (H = A_3(\varphi_3^{\cdot} + \omega^*\cos\varphi_2) + k = \text{const}) \quad (2.2)$$

The integral H follows immediately from the third equation of the system (1.2); ω^* is the angular velocity of precession, $\varphi_3^{\cdot}$ is the inherent angular velocity of the carrier, and φ_2 is the angle between axis ξ_3' and the axis of symmetry, with $\cos\varphi_2 = \alpha_{33}$.

It is not difficult to conclude that the motion defined above must take place on a circular orbit, and that the angular velocity ω^* of precession must coincide with the angular velocity of the mass center along the orbit.

The stipulated expression for the force function U yields a non-Keplerian value of the angular velocity. However, in the following we shall assume the mass center of the gyrostat is moving along a Keplerian orbit, defined by (1.1), provided that the terms dependent on the position of the body relative to the mass center are neglected. We can then assume that for a circular orbit, within the accepted degree of accuracy in the expansion of the force function, we have $\omega^2 = \mu / R^3$.

A comparison of the expression for the vector of the moment of gravitational forces with the derived formulas leads to expressions allowing us to determine the modes of regular precession which would satisfy all of the stated conditions

$$A_1\omega\cos\varphi_2\alpha_{32} + 3\omega(A_3 - A_1)\,\tau_{13}\tau_{12} = H\alpha_{32}, \quad A_1\,\omega\cos\varphi_2\alpha_{31} + 3\omega(A_3 - A_1)\,\tau_{13}\tau_{11} = H\alpha_{31}$$

We shall now write the expressions for the variables of our problem which would correspond to the three possible modes of regular precession of the gyrostat, by defining the position of the carrier in the Koenig system of coordinates by the usual Euler angles φ_1, φ_2, and φ_3.

First mode

$$\varphi_2 = \text{const}, \qquad H = A_1\omega\cos\varphi_2, \qquad \varphi_3^{\cdot} = \frac{(A_1 - A_3)\,\omega\cos\varphi_2 - k}{A_3}$$

$$\omega_1 = \omega\sin\varphi_2\sin\varphi_3, \quad \alpha_{31} = \sin\varphi_2\sin\varphi_3, \quad \tau_{21} = \cos\varphi_2\sin\varphi_3$$

$$\omega_2 = \omega\sin\varphi_2\cos\varphi_3, \quad \alpha_{32} = \sin\varphi_2\cos\varphi_3, \quad \tau_{22} = \cos\varphi_2\cos\varphi_3$$

$$\omega_3 = \varphi_3^{\cdot} + \omega\cos\varphi_2, \quad \alpha_{33} = \cos\varphi_2, \quad \tau_{23} = -\sin\varphi_2$$

$$\tau_{11} = \cos\varphi_3, \quad \tau_{12} = -\sin\varphi_3, \quad \tau_{13} = 0, \quad \varphi_3 = \varphi_3^{\cdot} t$$

Second mode

$$\varphi_2 = \text{const}, \qquad H = (4A_1 - 3A_3)\,\omega\cos\varphi_2, \qquad \varphi_3^{\cdot} = \frac{4(A_1 - A_3)\,\omega\cos\varphi_2 - k}{A_3}$$

$$\omega_1 = \omega\sin\varphi_2\sin\varphi_3, \quad \alpha_{31} = \sin\varphi_2\sin\varphi_3, \quad \tau_{21} = \cos\varphi_3$$

$$\omega_2 = \omega\sin\varphi_2\cos\varphi_3, \quad \alpha_{32} = \sin\varphi_2\cos\varphi_3, \quad \tau_{22} = -\sin\varphi_3$$

$$\omega_3 = \varphi_3^{\cdot} + \omega\cos\varphi_2, \quad \alpha_{33} = \cos\varphi_2, \quad \tau_{23} = 0$$

$$\tau_{11} = -\cos\varphi_2\sin\varphi_3, \quad \tau_{12} = -\cos\varphi_2\cos\varphi_3, \quad \tau_{13} = \sin\varphi_2, \quad \varphi_3 = \varphi_3^{\cdot} t$$

Third mode

$$\begin{array}{llll} & \varphi_2 = 0, & H = \text{const} & \\ \omega_1 = 0, & \alpha_{31} = 0, & \tau_{21} = \sin\varphi_3, & \tau_{11} = \cos\varphi_3 \\ \omega_2 = 0, & \alpha_{32} = 0, & \tau_{22} = \cos\varphi_3, & \tau_{12} = -\sin\varphi_3 \\ \omega_3 = \varphi_3^{\cdot} + \omega, & \alpha_{33} = 1, & \varkappa_{23} = 0, & \tau_{13} = 0 \end{array}$$

These modes of regular precession of a free gyrostat become, for $k = 0$, the modes of regular precession of a single rigid body [3].

3. The Liapunov analysis of the stability of these modes of regular precession of a gyrostat can be carried out by the method of Chetaev [4].

The problem considered here is essentially different from the general problem (see equations (1.1) to (1.3)) of motion of a free gyrostat in a central Newtonian field of forces, because of the assumption that in this case the motion of the system's mass center is Keplerian and its orbit circular. The equations of motion of the system consist of equations (1.2), supplemented by the Poisson's kinematic equations for the relative direction cosines. The system of differential equations will be complete, as the inner gyrostatic motion is defined by k = const. As the orbit is circular and the rotation of the mass center along it is regular, we have, in addition to simple geometric integrals and equation $A_3\omega_3 + k = H$ = const, Jacobi type integrals [5].

In the general case, in which $\mathbf{k} = \mathbf{k}\,(k_1, k_2, k_3)$, with the potential of internal forces denoted by Φ, and the constant angular velocity along the orbit of fixed radius of the mass center of the gyrostat denoted by ω, the Jacobi integral is of the form

$$\tfrac{1}{2}\,(A_1\omega_1^2 + A_2\omega_2^2 + A_3\omega_3^2) + k_1\omega_1 + k_2\omega_2 + k_3\omega_3 + T_g - \omega\,[(A_1\omega_1 + k_1)\,\alpha_{31} + (A_2\omega_2 + k_2)\,\alpha_{32} + (A_3\omega_3 + k_3)\,\alpha_{33}] = U + \Phi + h$$

where T_g is the kinetic energy of the gyrostatic elements in their motion relative to carrier S, and h is a constant energy.

Reverting to our problem, and noting that

$$A_1 = A_2 \neq A_3, \quad k_1 = k_2 = 0, \quad k_3 = k = \text{const}, \quad A_3\omega_3 + k = H, \quad \omega^2 = \mu/R^3$$

we obtain

$$A_1\,(\omega_1^2 + \omega_2^2) + A_3\omega_3^2 - 2\omega\,[A_1\,(\omega_1\alpha_{31} + \omega_2\alpha_{32}) + H\alpha_{33}] + 3\omega^2\,(A_3 - A_1)\,\tau_{13}^2 = \text{const} \tag{3.1}$$

It can be easily ascertained that the derived expression is in fact the first integral of the equations of motion.

For the analysis of stability of motion relative to the mass center it is convenient to introduce the angular velocity, relative to the orbital coordinate system, with projections

$$\omega_{r1} = \omega_1 - \omega\alpha_{31}, \qquad \omega_{r2} = \omega_2 - \omega\alpha_{32}, \qquad \omega_{r3} = \omega_3 - \omega\alpha_{33}$$

Integral (3.1) will then have the form

$$A_1\,(\omega_{r1}^2 + \omega_{r2}^2) + A_3\omega_{r3}^2 + (A_1 - A_3)\,\omega^2\alpha_{33}^2 - 2\omega k\alpha_{33} + 3\omega^2\,(A_3 - A_1)\,\tau_{13}^2 = \text{const}$$

also

$$A_3\omega_{r3} + A_3\omega\alpha_{33} + k = H$$

The analysis of stability of the first mode of regular precession with respect to variables ω_{r1}, ω_{r2}, ω_{r3}, α_{33}, and τ_{13}, will be made on the assumption that the orbital velocity ω, and the moment k remain unperturbed. Also, we assume that τ_{11}, τ_{12}, α_{31}, and α_{32} are cyclic.

For an unperturbed motion we have

$$\omega_{r1} = \omega_{r2} = 0, \qquad \omega_{r3} = \varphi_3^{\cdot} = \text{const}, \qquad \alpha_{33} = \cos\varphi_2, \qquad \tau_{13} = 0$$

In the case of a perturbed motion we shall denote these variables as follows

$$\omega_{r1}, \quad \omega_{r2}, \quad \omega_{r3} = \varphi_3^{\cdot} + \varepsilon, \quad \alpha_{33} = \cos\varphi_2 + \delta, \quad \tau_{13}$$

We write the first derived integrals of the equations of perturbed motion of the gyrostat

$$V_1 = A_1(\omega_{r1}^2 + \omega_{r2}^2) + A_3\varepsilon^2 + (A_1 - A_3)\omega^2\delta^2 + 3\omega^2(A_3 - A_1)\tau_{13}^2 + $$
$$+ 2\varphi_3^{\cdot}A_3\varepsilon + 2(A_1 - A_3)\omega^2\cos\varphi_2\delta - 2\omega k\delta = \text{const}$$
$$V_2 = A_3\varepsilon + A_3\omega\delta = \text{const}$$

We shall consider the following expression as a Liapunov function

$$W = V_1 - 2\varphi_3^{\cdot}V_2 + \lambda_1 V_2^2 = A_1(\omega_{r1}^2 + \omega_{r2}^2) + (A_3 + \lambda_1 A_3^2)\varepsilon^2 + $$
$$+ [(A_1 - A_3)\omega^2 + \lambda_1 A_3^2\omega^2]\delta^2 + \lambda_1 2A_3^2\omega\varepsilon\delta + 3\omega^2(A_3 - A_1)\tau_{13}^2 \tag{3.2}$$

Here

$$\lambda_1 = \text{const} > \frac{1}{A_1}\left(1 - \frac{A_1}{A_3}\right), \qquad \varphi_3^{\cdot} = \frac{(A_1 - A_3)\omega\cos\varphi_2 - k}{A_3}$$

Function W will have all the properties required of a Liapunov's function, when the last conditions above are fulfilled. But λ_1 can always be conveniently selected, and in the case of the first mode, the condition for $\varphi_3^{\cdot}$ is fulfilled by virtue of $H = A_1\omega\cos\varphi_2$, therefore, in accordance with Liapunov's theorem of stability, it is sufficient for the latter existence of the latter to have $A_3 > A_1$ in the indicated group of variables.

We shall now consider the stability of the second mode. For unperturbed motions in this mode we have

$$\omega_{r1} = \omega_{r2} = 0, \quad \omega_{r3} = \varphi_3^{\cdot}, \quad \alpha_{33} = \cos\varphi_2, \quad \tau_{23} = 0$$

For the perturbed mode we shall also assume that

$$\omega_{r1}, \quad \omega_{r2}, \omega_{r3} = \varphi_3^{\cdot} + \varepsilon, \quad \alpha_{33} = \cos\varphi_2 + \delta, \quad \tau_{23}$$

Let us write the integral (3.1) in the following form

$$A_1(\omega_{r1}^2 + \omega_{r2}^2) + A_3\omega_{r3}^2 + 4(A_1 - A_3)\omega^2\alpha_{33}^2 + 3\omega^2(A_1 - A_3)\tau_{23}^2 - 2\omega k\alpha_{33} = \text{const} \tag{3.3}$$

Since for the second mode we have $H = (4A_1 - 3A_3)\omega\cos\varphi_2$, therefore, the consideration of the following expression as the Liapunov function

$$W = V_1 - 2\frac{4(A_1 - A_3)\omega\cos\varphi_2 - V^2}{A_3} = $$
$$= A_1(\omega_{r1}^2 + \omega_{r2}^2) + A_3\varepsilon^2 + 4\omega^2(A_1 - A_3)\delta^2 + 3\omega^2(A_1 - A_3)\tau_{23}^2 \tag{3.4}$$

brings us to the immediate conclusion that for the second mode the sufficient condition of stability is $A_1 > A_3$.

For the third mode of unperturbed motion we have the following values of variables

$$\omega_{r1} = \omega_{r2} = 0, \quad \omega_{r3} = \varphi_3, \quad \tau_{13} = 0, \quad \alpha_{33} = 1, \quad \tau_{23} = 0$$

But this motion represents a particular case of the problem considered in [6]. Therefore, if $A_3 > A_1$, then the sufficient condition of stability is $A_3\varphi_3^{\cdot} > (A_1 - A_3)\omega - k$. If, however, $A_3 \leqslant A_1$, it is natural to use expression (3.4) as the Liapunov function with a prior change of variables in (3.3) to α_{31} and α_{32}, which in this mode have insignificant values. We then arrive at

$$A_3\varphi_3^{\cdot} > 4(A_1 - A_3)\omega - k$$

The conditions of stability of regular precession of a gyrostat established here coincide with those obtained in [3] for the stability of a single rigid body.

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